Cool + Cruel = Dual

and New Benchmarks for Sparse LWE

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(Sparse) LWE

LWE:

Given:

- $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$,
- $\mathbf{b} \equiv \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$, where $\|\mathbf{e}\|$ is small

Find:

- $\mathbf{s} \in \mathbb{Z}_q^n$

Sparse LWE:

- $\mathbf{s}$ has low Hamming weight (and small entries)
- relevant for FHE, but not for ML-KEM/Kyber

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Wenger, Saxena, Malhou, Thieu, Lauter

Benchmarking Attacks on Learning with Errors

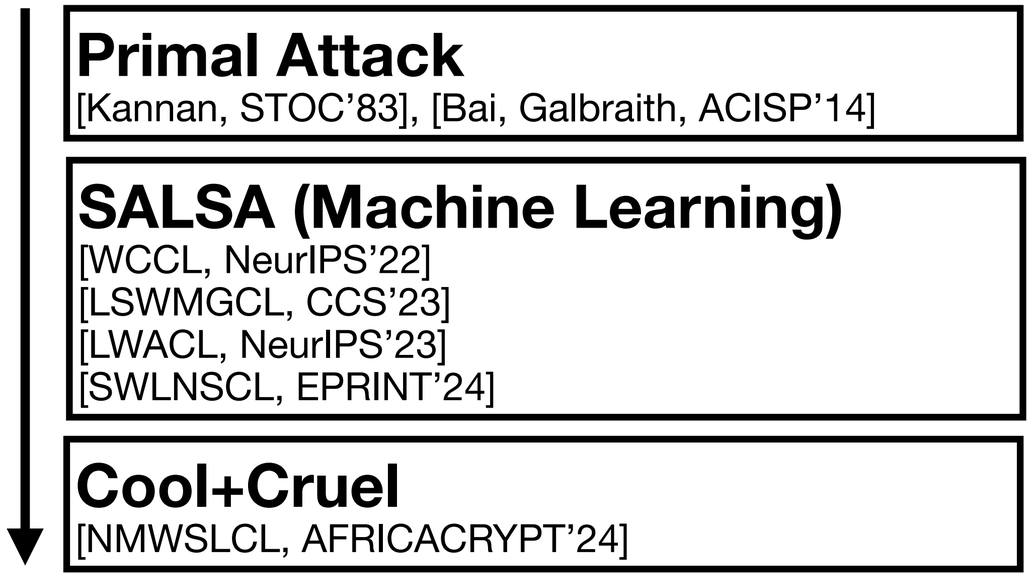
S&P'25

- benchmarks of Sparse LWE attacks on an industrial-scale cluster
- did not consider all relevant attacks

Attacks on Sparse LWE

[WSMTL, S&P'25] benchmarks suggest:

worst



best

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Primal Attack
[Kannan, STOC'83], [Bai, Galbraith, ACISP'14]

SALSA (Machine Learning)
[WCCL, NeurIPS'22]
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Cool+Cruel
[NMWSLCL, AFRICACRYPT'24]

best

We show:

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Dual Attack
[Aharonov, Regev, FOCS'05]
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Primal Attack
Drop+Solve [May, Silverman, CaLC'01]
Guess+Verify [Albrecht, Curtis, Wunderer, SAC'19]

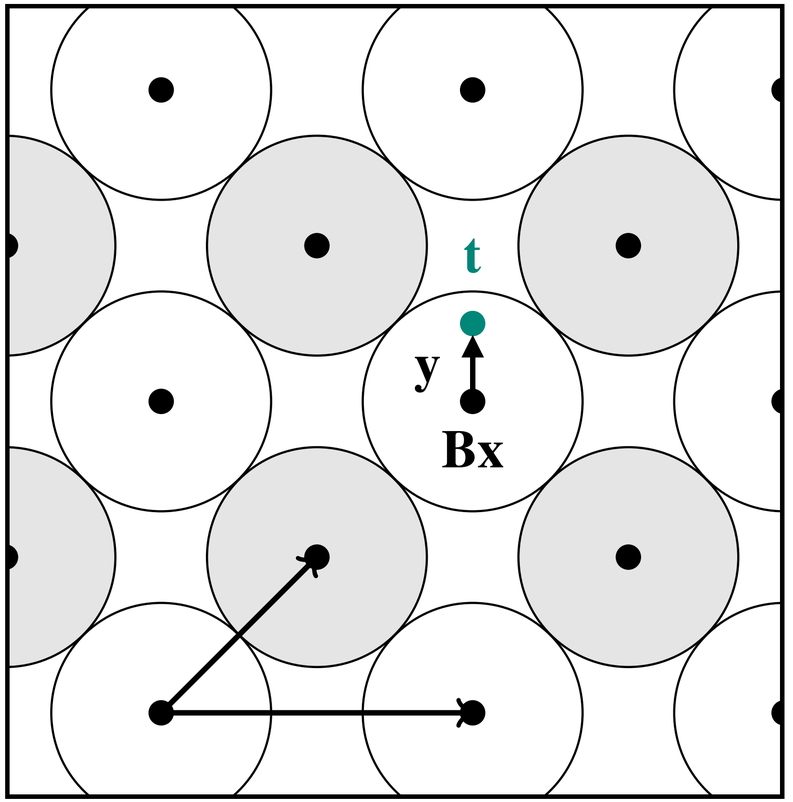
Bounded Distance Decoding (BDD)

Given:

- basis $\mathbf{B}$ of a lattice $\mathscr{L}(\mathbf{B}) := \mathbf{B} \cdot \mathbb{Z}^d$,
- $\mathbf{t} = \mathbf{B}\mathbf{x} + \mathbf{y}$, where $\mathbf{x} \in \mathbb{Z}^d$, and $\|\mathbf{y}\|$ is small

Find:

- $\mathbf{x} \in \mathbb{Z}^d$ minimizing $\|\mathbf{t} - \mathbf{B}\mathbf{x}\|$ (promised to be unique)



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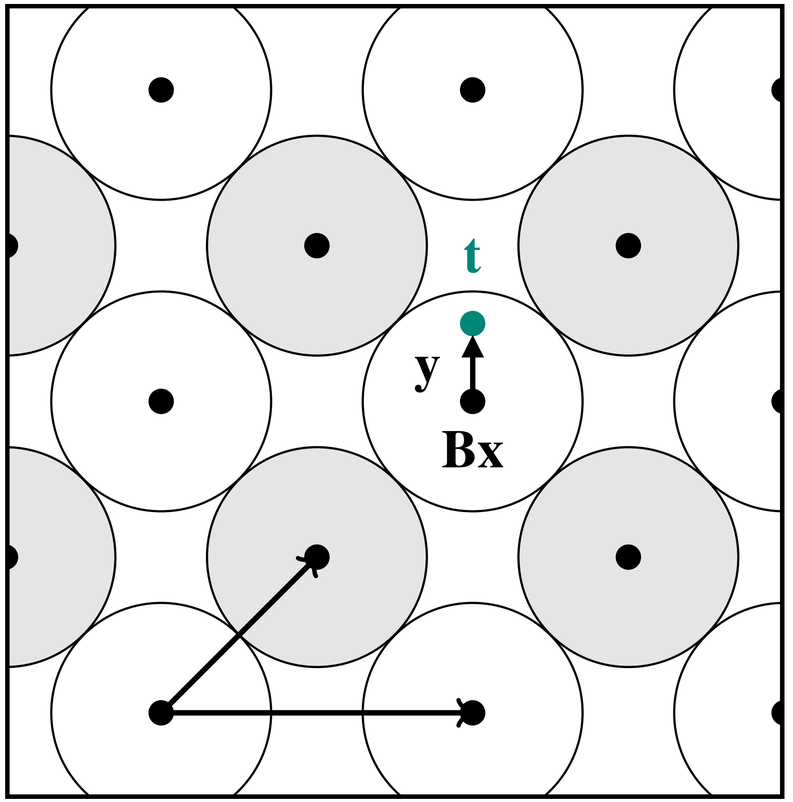

Sparse LWE $\mathbf{b} \equiv \mathbf{As} + \mathbf{e} \mod q$:

$$\mathcal{L}(\mathbf{B}) = \left\{(\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{Z}^n \times \mathbb{Z}^m \mid \mathbf{v}_2 \equiv -\mathbf{A}\mathbf{v}_1 \mod q\right\}$$

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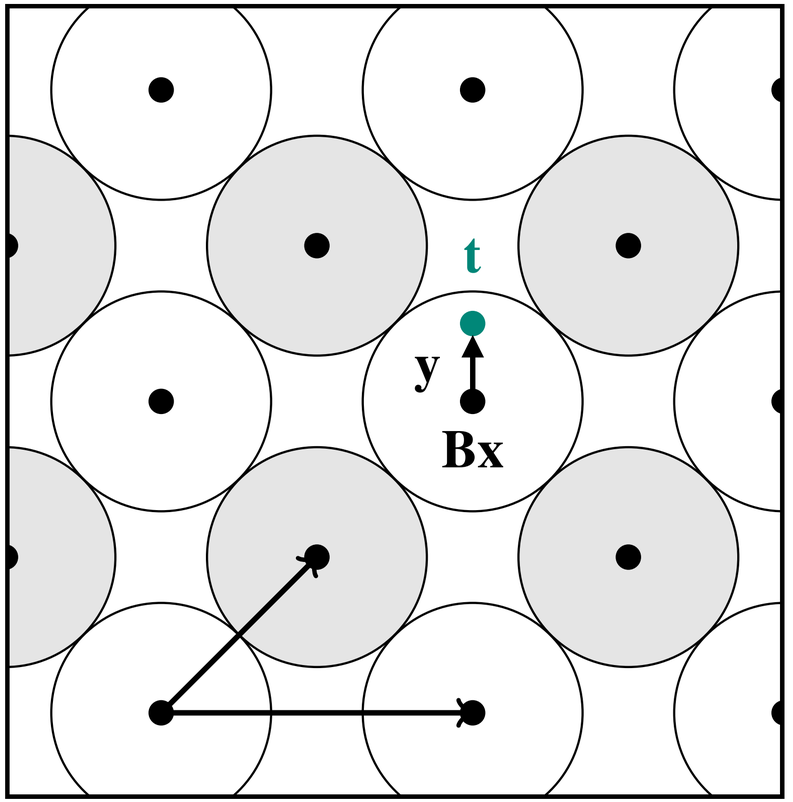

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$$\mathbf{Bx} = (-\mathbf{s}, \mathbf{b} - \mathbf{e}), \qquad \mathbf{y} = (\mathbf{s}, \mathbf{e}), \qquad \mathbf{t} = (0^n, \mathbf{b})$$

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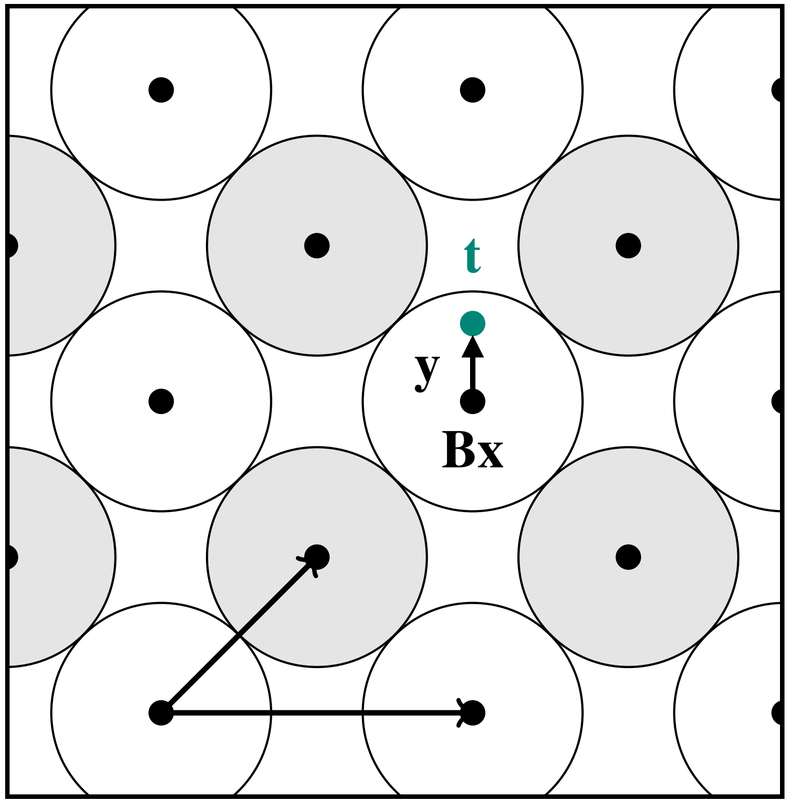

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→ relatively few candidates for $\mathbf{Bx}$

☞ hybrid attacks: lattice reduction + enumeration

Hybrid Attacks

Step 1:

- Split BDD instance $\mathbf{t} = \mathbf{B}_0\mathbf{x}_0 + \mathbf{B}_1\mathbf{x}_1 + \mathbf{y}$
- Enumerate list of $\mathbf{t}^{(i)} := \mathbf{t} - \mathbf{B}_1\mathbf{x}_1^{(i)}$

$$\mathbf{B} = [\mathbf{B}_0, \mathbf{B}_1], \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$$

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- Apply BDD solver to every $\mathbf{t}^{(i)}$
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Hybrid Attacks

Guess+Verify

Cool+Cruel

+ information set decoding

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Cool+Cruel < Guess+Verify

Sparse LWE with $n = 512$, $q \approx 2^{28}$

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[WSMTL, S&P'25]

Primal Attack 1 CPU
Cool+Cruel **lattice reduction** 151 CPUs
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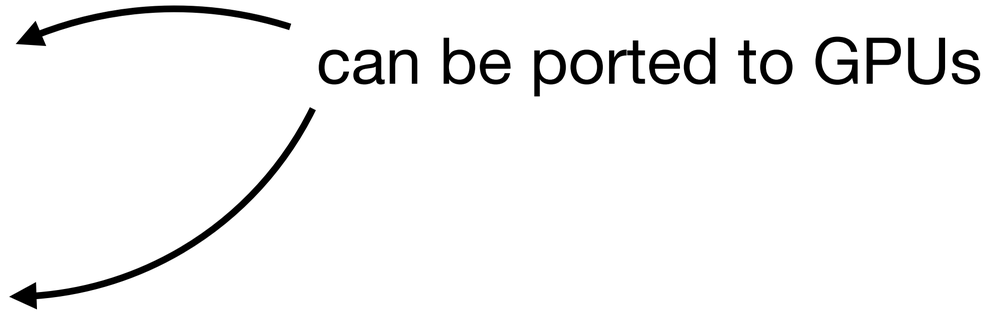

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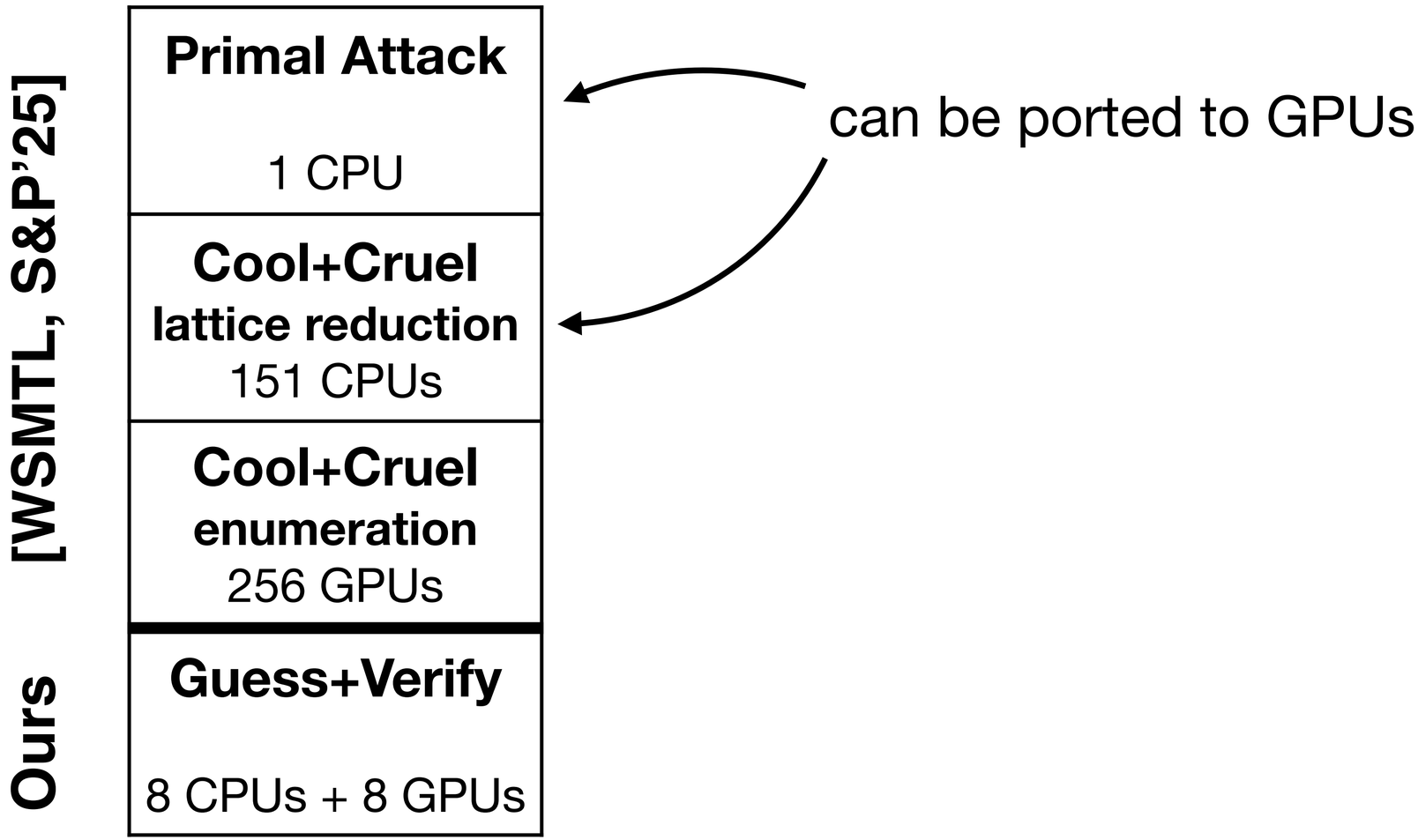

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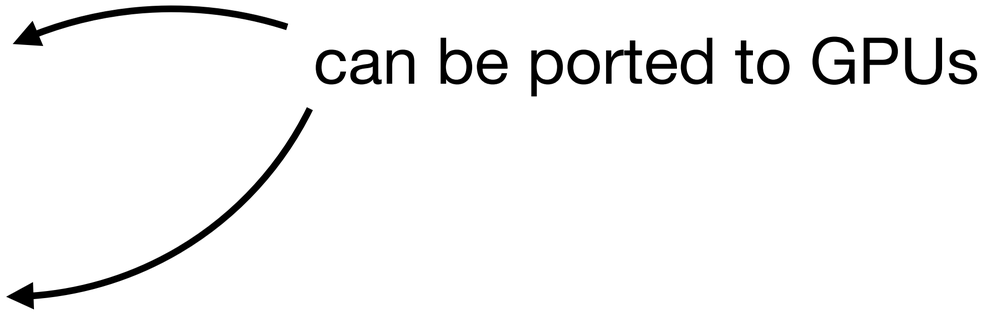
Ours

Guess+Verify

8 CPUs + 8 GPUs



Hamming weight	**20**	**21**	**25**
Cool+Cruel enumeration min. GPU-hours	**38**	**141**	**10642**
Guess+Verify avg. GPU-hours	**23**	**38**	**164**

Conclusion

Summary:

- Drop+Solve/Guess+Verify Primal > Cool+Cruel = Dual

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ia.cr/2025/1002

github.com/ludopulles/eprint-2025-1002

Appendix

Cool+Cruel

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Output: First n_u coordinates of $\mathbf{s}^T = (\mathbf{s}_u, \mathbf{s}_r)^T \in \mathbb{Z}^{n_u} \times \mathbb{Z}^{n_r}$

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$$\mathbf{D} := \begin{bmatrix} q\mathbf{I}_n & \mathbf{A}^T \\ & \mathbf{I}_m \end{bmatrix} \xrightarrow[\text{on submatrices of } \mathbf{D}]{\text{lattice reduction}} \text{many shortish vectors } \mathbf{w} \in \mathscr{L}(\mathbf{D}) \xrightarrow[\text{has special shape}]{\text{observe that } \mathbf{w}} \mathbf{w} = \begin{pmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} \begin{matrix} \} n_u \\ \} n_r \\ \} m \end{matrix}$$

short

$$\mathbf{A}_u^{\text{red}} := \begin{bmatrix} \vdots \\ \mathbf{w}_0^T \\ \vdots \end{bmatrix} \qquad \mathbf{A}_r^{\text{red}} := \begin{bmatrix} \vdots \\ \mathbf{w}_1^T \\ \vdots \end{bmatrix} \qquad \mathbf{b}^{\text{red}} := \begin{pmatrix} \vdots \\ \langle \mathbf{w}_2, \mathbf{b} \rangle \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} \vdots \\ \mathbf{w}_2^T(\mathbf{A}\mathbf{s} + \mathbf{e}) \\ \vdots \end{pmatrix} \equiv \mathbf{A}_u^{\text{red}}\mathbf{s}_u + \mathbf{A}_r^{\text{red}}\mathbf{s}_r + \begin{pmatrix} \vdots \\ \langle \mathbf{w}_2, \mathbf{e} \rangle \\ \vdots \end{pmatrix} \mod q$$

$$\mathbf{w}_2^T\mathbf{A} \equiv (\mathbf{w}_0, \mathbf{w}_1)^T \mod q$$

Enumerate candidates $\mathbf{s}_u^{(i)}$ for $\mathbf{s}_u$. Take the one that makes $\mathbf{b}^{\text{red}} - \mathbf{A}_u^{\text{red}}\mathbf{s}_u^{(i)}$ short (mod q).

Cool+Cruel

Input: LWE instance $\mathbf{b} \equiv \mathbf{A}\mathbf{s} + \mathbf{e} \mod q$

Output: First n_u coordinates of $\mathbf{s}^T = (\mathbf{s}_u, \mathbf{s}_r)^T \in \mathbb{Z}^{n_u} \times \mathbb{Z}^{n_r}$

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Dual Attack:

- Split BDD instance $\mathbf{t} = \mathbf{B}_0\mathbf{x}_0 + \mathbf{B}_1\mathbf{x}_1 + \mathbf{y}$
- Compute short vectors $\mathbf{w}' \in \mathscr{L}(\mathbf{B}_0)^*$
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LWE:

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