Approximate Divisor Multiples
Factoring with Only a Third of the Secret CRT-Exponents

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RSA Keys

RSA public key:

$$\boxed{N = pq}$$

$$\boxed{e}$$

RSA private key:

$$\boxed{N = pq}$$

$$\boxed{d = e^{-1} \mod (p-1)(q-1)}$$

CRT-RSA Keys

CRT-RSA public key:

$$\boxed{N = pq}$$

$$\boxed{e}$$

CRT-RSA private key:

$$\boxed{N = pq}$$

$$\boxed{d = e^{-1} \mod (p-1)(q-1)}$$

$$\boxed{p}$$

$$\boxed{q}$$

$$\boxed{d_p = d \mod (p-1)}$$

$$\boxed{d_q = d \mod (q-1)}$$

$$\boxed{q_{inv} = q^{-1} \mod p}$$

CRT-RSA Keys

CRT-RSA public key:

$$\boxed{N = pq}$$

$$\boxed{e}$$

CRT-RSA private key:

$$\cancel{N = pq}$$

$$\cancel{d = e^{-1} \bmod (p-1)(q-1)}$$

$$\boxed{p}$$

$$\cancel{q}$$

$$\cancel{d_p = d \bmod (p-1)}$$

$$\cancel{d_q = d \bmod (q-1)}$$

$$\cancel{q_{inv} = q^{-1} \bmod p}$$

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$$\cancel{d_p = d \bmod (p-1)}$$

$$\cancel{d_q = d \bmod (q-1)}$$

$$\cancel{q_{inv} = q^{-1} \bmod p}$$

Partial Key Exposure Attacks

Theorem (Coppersmith EC'96)

Given half of the bits of p, we can factor N in polynomial time.

Coppersmith's attack is efficient:

Bit-size of N	Runtime on a laptop
1024	$\approx$ 2min
2048	$\approx$ 6min
4096	$\approx$ 24min

Coppersmith's attack is practical:

- [BCC+13] breaks $\approx$ 80 smart cards.
- [NSS+17] breaks $\approx 10^7$ smart cards.

Theorem (Boneh, Durfee, Frankel AC'98)

Suppose $e = \mathcal{O}(\log N)$. Given a quarter of the bits of d, we can factor N in polynomial time.

Theorem (Blömer, May CRYPTO'03)

Suppose $e = \mathcal{O}(\log N)$. Given half of the bits of d_p, we can factor N in polynomial time.

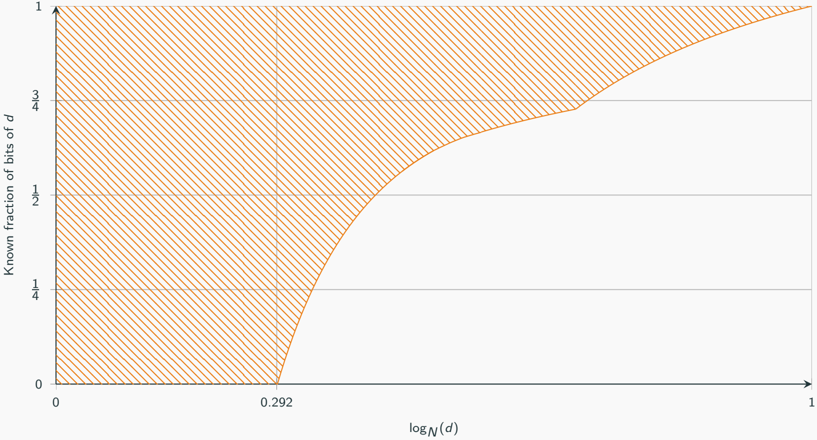
- For n-bit N, these attacks require $\frac{n}{4}$ bits.

☞ $p \approx N^{1/2}$, $d \approx N$, $d_p \approx N^{1/2}$.

Partial Key Exposure Attacks

Theorem (Ernst, Jochemsz, May, de Weger EC'05; Aono PKC'09; Takayasu, Kunihiro SAC'14)

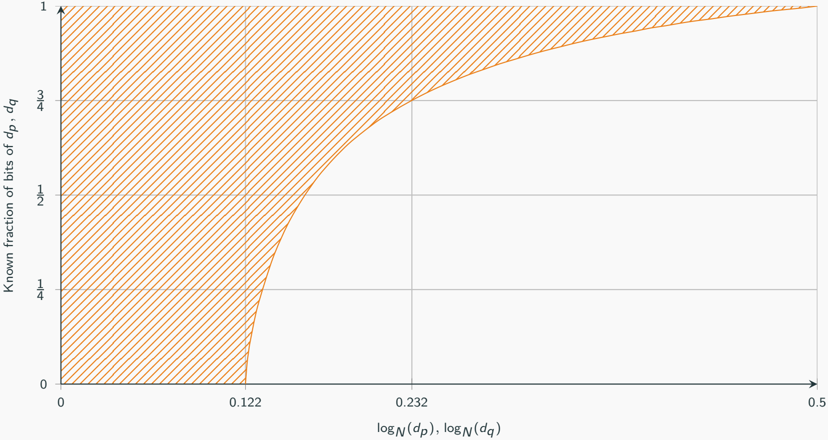
Suppose $e = \mathcal{O}(N)$. The smaller d, the less bits of d we have to know to factor N in polynomial time (assuming a well-established heuristic).



Partial Key Exposure Attacks

Theorem (May, N., Sarkar AC'21)

Suppose $e = \mathcal{O}(N)$. The smaller d_p, d_q, the less bits of d_p, d_q we have to know to factor N in polynomial time (assuming a well-established heuristic).



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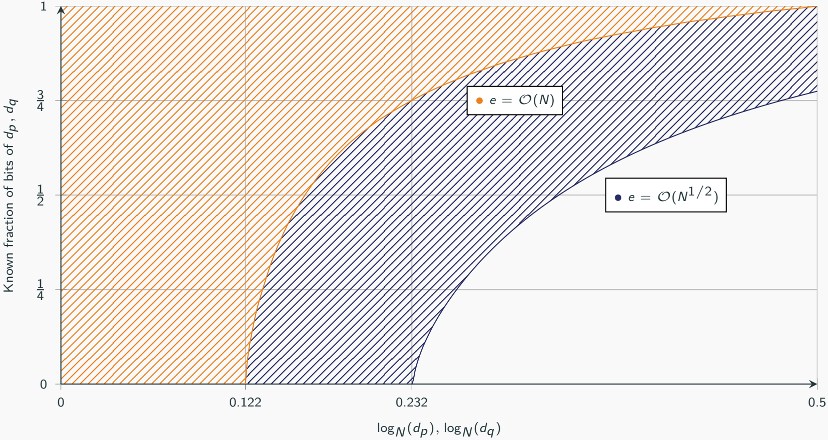


Partial Key Exposure Attacks

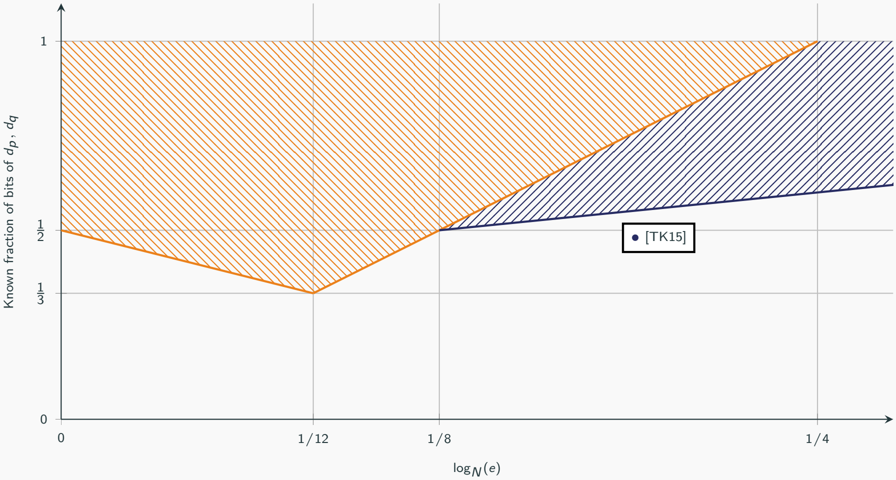
Partial Key Exposure attacks in a nutshell:

- The smaller e, d, d_p, d_q, the less bits we have to know to factor N in polynomial time.

Our result:

- New Partial Key Exposure attack for exposed d_p, d_q and small(-ish) $e < N^{1/4}$.
- Surprsing behaviour for $e \leq N^{1/12}$:
  The larger e, the less bits we have to know to factor N in polynomial time.

Our Partial Key Exposure Attack



Why Our Attack Behaves Differently

The usual strategy for RSA Partial Key Exposure attacks:

Model problem as system of polynomial equations

$$\left|\begin{matrix} f_1(x_1, \dots, x_k) = 0 \\ \vdots \\ f_n(x_1, \dots, x_k) = 0 \end{matrix}\right|.$$

$\downarrow$

Apply Coppersmith's method.

Our new strategy:

Model problem as system of polynomial equations

$$\left|\begin{matrix} f_1(x_1, \dots, x_k) = 0 \\ \vdots \\ f_n(x_1, \dots, x_k) = 0 \end{matrix}\right|.$$

$\downarrow$

Compute partial solution in few variables.

$\downarrow$

Apply Coppersmith's method.

Step 1: Compute Partial Solution in Few Variables

There exist $k, \ell \in \mathbb{N}$, such that

$$
\begin{aligned}
ed_p &= 1 + k(p-1), \\
ed_q &= 1 + \ell(q-1).
\end{aligned}
$$

Step 1: Compute Partial Solution in Few Variables

There exist $k, \ell \in \mathbb{N}$, such that

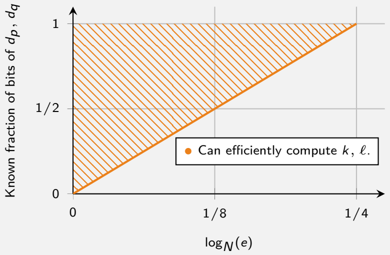
$$ed_p = 1 + k(p-1),$$
$$ed_q = 1 + \ell(q-1).$$

Question

How difficult is computing k, ℓ?

- Folklore: If $e = \mathcal{O}(\log N)$, then brute-force search runs in polynomial time.
- [GHM05]: If $e \geq N^{1/4}$, then as hard as factoring.

Our result:



Step 1: Compute Partial Solution in Few Variables

There exist $k, \ell \in \mathbb{N}$, such that

$$ed_p^{\text{MSB}} \approx e(d_p^{\text{MSB}} || d_p^{\text{LSB}}) = 1 + k(p-1) \approx kp,$$
$$ed_q^{\text{MSB}} \approx e(d_q^{\text{MSB}} || d_q^{\text{LSB}}) = 1 + \ell(q-1) \approx \ell q.$$

$$\implies e^2 d_p^{\text{MSB}} d_q^{\text{MSB}} \approx k\ell N.$$
$$\implies \frac{e^2 d_p^{\text{MSB}} d_q^{\text{MSB}}}{N} \approx k\ell.$$

Lemma

If $d_p^{\text{MSB}}, d_q^{\text{MSB}} > e^2$, then $\left\lceil \frac{e^2 d_p^{\text{MSB}} d_q^{\text{MSB}}}{N} \right\rceil = k\ell$.

Lemma

We can split $k\ell$ into k and ℓ in time $\mathcal{O}(\log^2 N)$.

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How difficult is computing k, ℓ?

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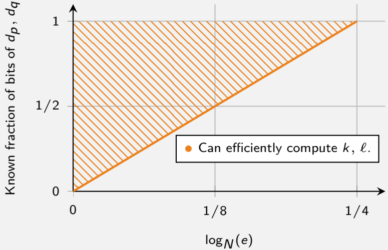

Step 2: Apply Coppersmith's Method

Problem (Approximate GCD Problem)

Given:

- $N_0 = q_0 s$
- $N_1 \approx q_1 s$

Find:

- s

Theorem (Howgrave-Graham CaLC'01)

If $s \geq N_0^{\beta}$, $\beta \in [0, 1]$ and $|N_1 - q_1 s| < N_0^{\beta^2}$, then we can compute s in polynomial time.

- Algorithm is based on Coppersmith's method.

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Our attack scenario

Given:

- $N = qp$
- $ed_p^{\mathrm{MSB}} \approx kp$
- k

Find:

- p

Step 2: Apply Coppersmith's Method

Problem (Approximate GCD Multiple Problem)

Given:

- $N_0 = q_0 s$
- $N_1 \approx q_1 s$
- q_1

Find:

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Our attack scenario

Given:

- $N = qp$
- $ed_p^{\mathrm{MSB}} \approx kp$
- k

Find:

- p

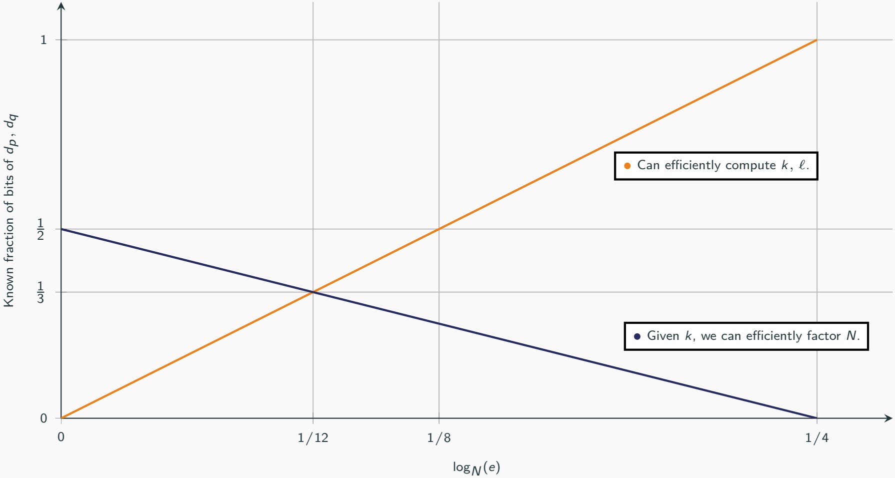
Corollary

Given k and d_p^{MSB} with

$$d_p^{\mathrm{MSB}} > \frac{N^{1/4}}{e},$$

we can factor N in polynomial time.

Putting Everything Together



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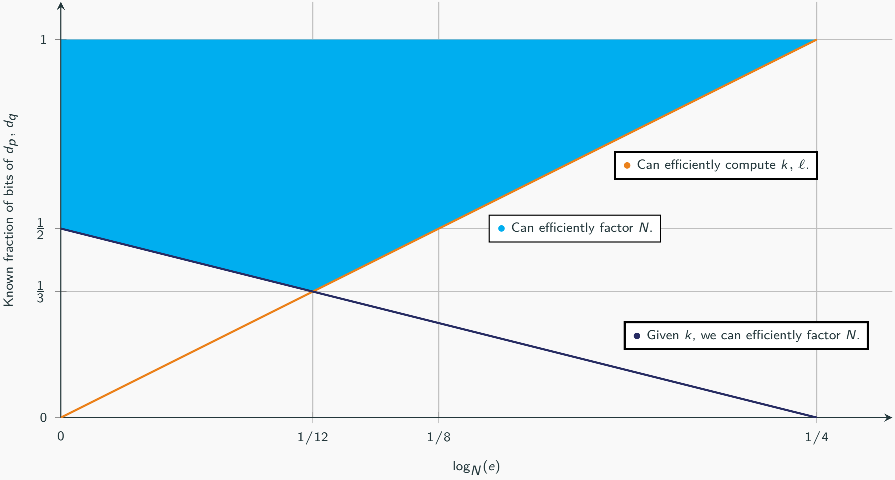

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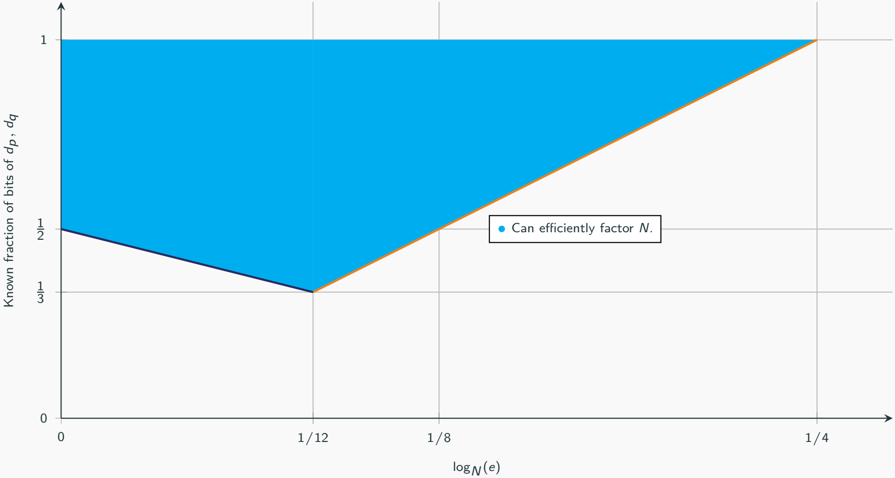
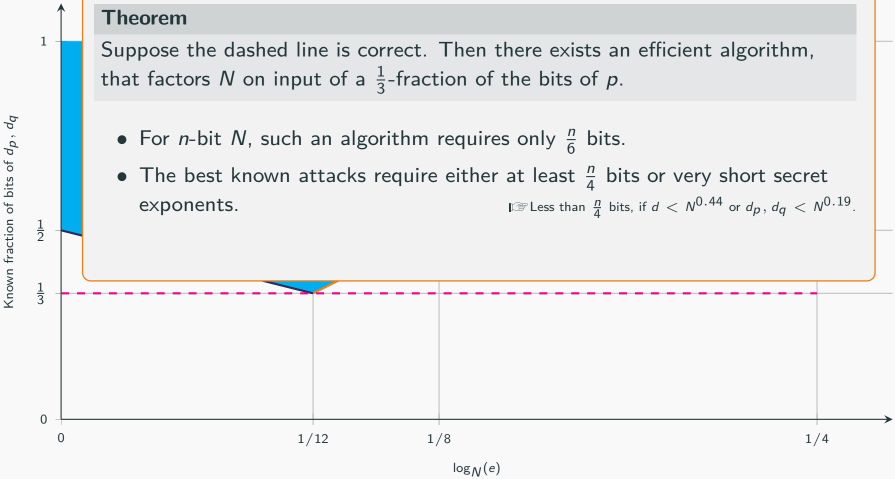


Putting Everything Together

Theorem

Suppose the dashed line is correct. Then there exists an efficient algorithm, that factors N on input of a $\frac{1}{3}$-fraction of the bits of p.

- For n-bit N, such an algorithm requires only $\frac{n}{6}$ bits.
- The best known attacks require either at least $\frac{n}{4}$ bits or very short secret exponents. ☞Less than $\frac{n}{4}$ bits, if $d < N^{0.44}$ or $d_p, d_q < N^{0.19}$.



Conclusion and Open Problems

Conclusion:

- Previously known Partial Key Exposure attacks work the better, the smaller e, d, d_p, d_q.
- First Partial Key Exposure attack on RSA, with a different behavior.
- Works best for $e \approx N^{1/12}$.
- Take-away: Do not apply Coppersmith's method directly to your system of polynomial equations. Check first, if you can eliminate some variables by different means.

Open Problems:

- Which size of e should we use in practice?
- Is $e \approx N^{1/12}$ the least secure?
- Does our algorithm for the AGCD-Multiple-Problem have implications for the AGCD-Problem?

☞Security of "FHE over the integers" and LWE?

Comparison Between Partial Key Exposure Attacks

Exposed variable	Constraint	Required bits
p	-	$\frac{n}{4}$
d	$e = \mathcal{O}(\log N)$	$\frac{n}{4}$
d_p	$e = \mathcal{O}(\log N)$	$\frac{n}{4}$
d	$d < N^{0.44}$	$< \frac{n}{4}$
d	$d < N^{0.36}$	$< \frac{n}{8}$
d	$d < N^{0.29}$	0
d_p, d_q	$d_p, d_q < N^{0.29}$	$< 2 \times \frac{n}{4}$
d_p, d_q	$d_p, d_q < N^{0.19}$	$< 2 \times \frac{n}{8}$
d_p, d_q	$d_p, d_q < N^{0.12}$	0
d_p, d_q	$e \leq N^{1/8}$	$\leq 2 \times \frac{n}{4}$
d_p, d_q	$e \approx N^{1/12}$	$2 \times \frac{n}{6}$